

Quantum Properties of Periodic Instantons on a Circle

A. V. SHURGAIA¹

Department of Theoretical Physics
A. Razmadze Mathematical Institute
1 M. Alexidze str. 0193 Tbilisi
Georgia

Abstract

Quantum properties of a (1+1)-dimensional scalar theory on a cylinder with a compact spatial part, namely, $0 \leq x \leq L$, are considered. In particular, quantum theory around the classical periodic field configurations is studied and the lifetime of a quantum periodic instanton is estimated.

1. Periodic field configurations known as the periodic instantons have been for many years in the field of view of researchers in view of their important role in the process of quantum tunneling and the associated phase transitions. The attractiveness of such field objects is that, depending on the total energy they interpolate between the vacuum and saddle point states (sphalerons) and thus have the ability to be responsible for the phase transition from the classical processes at high energies to quantum tunneling ones at low energies. The complexity of actual physical theories does not allow to obtain exact results, hence lower dimensional models which make it possible to perform some analytical calculations are of interest. From this point of view (1+1)-dimensional scalar theory is subjected to intense study. Along with stable field configurations of finite energy more recently unstable configurations like bounces [1], sphalerons[2, 3], periodic instantons [4, 5] have been investigated. Recently, interest has grown in configurations on compact space and configurations like (anti)periodic (called also twisted fields) are in addition investigated[6]-[9]. In the cited papers the exact solutions of classical equations of motion on a circle, as well as the equation of fluctuations are investigated and regularization of the energy in one-loop approximation is carried out. In the present paper we are interested in periodic classical field configurations and their quantum properties in two-dimensional scalar theory, the spatial part of which is a circle with circumference L . There is an interesting interpretation of nontrivial classical field configurations for finite L , provided, that the potential of the model we explore has two distinct vacua separated by a finite potential barrier. This allows to consider the process of creating of kink-antikink pair at some energy. They

¹e-mail: avsh@rmi.acnet.ge

propagate along the circle in opposite directions and annihilate at the meeting points leaving the field in the other vacuum. The largest distance between kink and antikink corresponds to the state at the top of a potential barrier - that is the saddle point of the energy functional. With increasing L unstable configurations of many kink-antikink states can be appeared. In the present paper we start from the quantum theory in Schrödinger representation. We use the symmetry properties of the theory and construct a perturbation theory using the collective coordinates method (see Ref.[10] and references therein) around the classical field configuration. We calculate the quantum energy levels and estimate the lifetime of the system.

2. Thus, we consider the system with the lagrangian density

$$\mathcal{L} = \frac{1}{2} \frac{\partial \phi(t, x)}{\partial x_\mu} - U(\phi(t, x), g) \quad (1)$$

with the potential satisfying

$$U(\phi(t, x), g) = \frac{1}{g^2} U(g\phi(t, x), 1)$$

in which $0 \leq x \leq L$. We work in the Schrödinger representation and look for solutions of the equation:

$$(H - E)\Psi(\phi(x), \pi(x)) = 0. \quad (2)$$

The corresponding quantum Hamiltonian is

$$H = \int_0^L dx \left\{ \frac{1}{2} \pi(x)^2 + \frac{1}{2} \left(\frac{\partial \phi(x)}{\partial x} \right)^2 + U(\phi(x), g) \right\} \quad (3)$$

subject to the following commutation relation:

$$[\phi(x), \pi(y)] = i\delta(x - y). \quad (4)$$

We now develop a perturbation theory for the double-well potential $U(\phi(x), g)$

$$U(\phi(x), g) = \frac{\mu^2}{2g^2} (\phi(x)^2 - g^2)^2. \quad (5)$$

The system possesses a translational invariance and corresponding conserved quantity is the momentum of the system:

$$P = - \int_0^L dx \frac{\partial \phi(x)}{\partial x} \pi(x) \quad (6)$$

Let $\phi_0(x)$ be a c-number and consider the following transformation of the field $\phi(x)$:

$$\phi(x) = g\phi_0(x - q) + \Phi(x - q). \quad (7)$$

The parameter q , being called a collective coordinate, together with the new field $\Phi(x)$ form now a new set of variables (operators) of the system: $\{\phi(x), \pi(x)\} \rightarrow \{q, p_q, \Phi(x), \tilde{\Pi}(x)\}$. Thereby the phase space has been expanded and in order to keep the number of independent variables unchanged the additional condition must be imposed (actually this is a gauge condition):

$$\int_0^L dx N(x) \Phi(x) = 0, \quad (8)$$

in which $N(x)$ is an arbitrary function. Really we need one more additional condition, but as we will see below this condition will appear as a constraint in the extended space. One can always choose $N(x)$ such that the relation

$$\int_0^L dx N(x) \frac{\partial \phi_0(x)}{\partial x} = 1. \quad (9)$$

holds. The operator $\pi(x)$ is considered as a functional derivative $-i\delta/\delta\phi(x)$ in a Hilbert space spanned by the vectors $|\Psi\rangle$ and hence should be expressed through the set of new variables. For this we need to define the projection operator $A(x, y) = \delta(x - y) - \frac{\partial \phi_0(x)}{\partial x} N(y)$, with the properties

$$\int_0^L dy A(x, y) \frac{\partial \phi_0(y)}{\partial x} = \int_0^L dx N(x) A(x, y) = 0.$$

Now it is easy to show that the operator $\pi(x)$ can be rewritten as follows:

$$\pi(x) = \Pi(x - q) - \frac{N(x - q)}{g(1 + \frac{1}{g}F)} \{p_q + M(\Phi, \Pi)\}, \quad (10)$$

in which the quantities F and $M(\Phi, \Pi)$ are defined by the following equalities:

$$F = \int_0^L dx N(x) \frac{\partial \Phi(x)}{\partial x}, \quad M(\Phi, \Pi) = \int_0^L dx \frac{\partial \Phi(x)}{\partial x} \Pi(x). \quad (11)$$

Besides the operator $\Pi(x)$, determined as

$$\Pi(x) = \int_0^L dy A(y, x) \frac{\delta}{i\delta\Phi(y)}, \quad (12)$$

satisfies the condition

$$\int_0^L dx \frac{\partial \phi_0(x)}{\partial x} \Pi(x) = 0, \quad (13)$$

that is the constraint of theory (that we have mentioned above) being considered as a system with constraints. The operators q and $p_q = \frac{\partial}{i\partial q}$ are conjugate to each other such that $[q, p_q] = i$. Thus all extra degrees of freedom are now fixed. The operator $\Pi(x)$ obeys the following commutation relation:

$$[\Phi(x), \Pi(y)] = iA(x, y). \quad (14)$$

One can now check by substituting (10) into (6) that the momentum operator of the system coincides with p_q and consequently $P = p_q$. On substituting (7) and (10) into (3) one obtains the Hamiltonian in the new phase space. We see, that $\pi(x)$ depends on q . In order to obtain the Hamiltonian we have to integrate over x in the range from 0 to L . Shifting the integration variable x changes the limits of integration from $-q$ to $L - q$. Nevertheless there is no explicit dependence of the Hamiltonian from q . Taking into consideration the periodicity of the phase space variables and the Leibnitz integral rule, it is easy to verify that $\frac{dH}{dq} = 0$. This allows us to extract the q -dependence of the wave function of the system $\Psi(\Phi(x), q)$. Before doing so we need to transform the wave function, indeed

$$\Psi(\Phi, q) = e^{ig \int_0^L dx s(x) \Phi(x)} \Psi'(\Phi(x), q). \quad (15)$$

which means that the momentum operator $\Pi(x)$ has to be replaced by $gs(x) + \Phi(x)$. The c-number $s(x)$ is subject to the same constraint as the operator $\Pi(x)$, namely $\int_0^L dx \frac{\partial \phi_0(x)}{\partial x} s(x) = 0$. If this condition is not being fulfilled then one defines with the help of the projection operator a new quantity $s'(x)$ that obeys it. Thus the equality (10) may be rewritten as follows:

$$\pi(x) = \Pi(x - q) - \frac{N(x - q)}{g(1 + \frac{1}{g}F)} \{p_q + gM(\Phi, s) + M(\Phi, \Pi)\}. \quad (16)$$

3. After all these one can start to solve the Schrödinger equation $(H - E)\Psi'(\Phi(x), q) = 0$. As we have seen above the Hamiltonian H does not explicitly depend on q and therefore one may factorize the wave function $\Psi'(\Phi(x), q)$ as follows:

$$\Psi'(\Phi(x), q) = e^{ig^2 I q} \Psi''(\Phi(x)). \quad (17)$$

Thus the operator p_q should be replaced by the c-number $g^2 I$. We will solve the Schrödinger equation by using a perturbation theory and consequently we need now to expand the hamiltonian in series in inverse powers of g . Energy and the wave function are also to expand in appropriate series:

$$H = g^2 H_0 + g H_1 + H_2 + g^{-1} H_3 + \dots, \quad (18)$$

$$E = g^2 E_0 + g E_1 + E_2 + g^{-1} E_3 + \dots, \quad (19)$$

$$\Psi''(\Phi(x)) = \Psi_0 + g^{-1} \Psi_1 + \dots \quad (20)$$

So we should solve the system of equations:

$$(H_0 - E_0)\Psi_0 = 0, \quad (21)$$

$$(H_0 - E_0)\Psi_1 + (H_1 - E_1)\Psi_0 = 0, \quad (22)$$

$$((H_0 - E_0)\Psi_2 + (H_1 - E_1)\Psi_1 + (H_2 - E_2)\Psi_0 = 0. \quad (23)$$

.....

Leading term of the Hamiltonian does not contain field operators and therefore the equation in corresponding approach $(H_0 - E_0)\Psi_0 = 0$ is obeyed identically

if

$$E_0 = H_0 = \int_0^L dx \left\{ \frac{1}{2} \left(\frac{\partial \phi_0(x)}{\partial x} \right)^2 + U(\phi(x), 1) + \frac{1}{2} (s(x) - IN(x))^2 \right\}. \quad (24)$$

The next approximation of the system of equations we consider is:

$$(\Psi_0, (H_1 - E_1)\Psi_0) = 0, \quad (25)$$

which is linear in field operators. The regularity of the function Ψ_0 requires H_1 and E_1 to be identically zero. This is accomplished by using the additional conditions imposed on the field operators, and if required

$$s(x) - IN(x) = -v \frac{\partial \phi_0(x)}{\partial x}, \quad (26)$$

and

$$-(1 - v^2) \frac{\partial^2 \phi_0(x)}{\partial x^2} + U'(\phi_0(x), 1) = 0 \quad (27)$$

This is a pure classical equation of motion, the solution of which under periodic boundary condition for the potential U of interest is:

$$\phi_0(x) = \sqrt{\frac{2k^2}{1+k^2}} \text{sn}\left(\sqrt{\frac{2}{1+k^2}} \frac{\mu x}{\sqrt{1-v^2}}, k\right), \quad (28)$$

$$(29)$$

in which $0 \leq k \leq 1$ is the modulus of elliptic integrals. The periodicity condition $\phi_0(x + L) = \phi_0(x)$ implies that

$$L = 4nK(k) \sqrt{\frac{1+k^2}{2}} \frac{\sqrt{1-v^2}}{\mu}, \quad (30)$$

$$(31)$$

with n integer. So, there are critical values of the quantity L , at which there are bifurcations of the stationary points of the energy E_0 . These solutions interpolate between the stable field configurations as k^2 tends to 1 (with L approaching the ∞) and the unstable ones (at the top of the potential barrier) for $k^2 \rightarrow 0$, latter being called sphalerons (see Ref.[4],[5] for details).

Let us now turn to the zero energy E_0 , that takes the following form:

$$E_0 = \int_0^L dx \left\{ \frac{1}{2} (1 + v^2) \left(\frac{\partial \phi_0(x)}{\partial x} \right)^2 + U(\phi(x), 1) \right\} \quad (32)$$

and is for (5):

$$E_0 = \frac{n\sqrt{2}\mu [8(1+k^2)E(k) - [(1-k^2)(5+3k^2) + 3(1-k^2)v^2]K(k)]}{3(1+k)^{3/2}\sqrt{1-v^2}}, \quad (33)$$

which for $v = 0$ is consistent with the expression obtained in [4]. Using the equality (26) and the condition that is imposed on $s(x)$ one can obtain for the momentum I :

$$I = \frac{4n\sqrt{2}\mu[(k^2 - 1)K(k) + E(k)]}{3\sqrt{(1 - v^2)}(1 + k^2)^{3/2}}v. \quad (34)$$

It is easy to verify that v is a velocity of the center-of-mass of the system.

4. Let us now proceed with the study of quantum correction to the ground state energy, for which the equation

$$(H_2 - E_2)\Psi_0 = 0 \quad (35)$$

has to be solved. The operator H_2 is quadratic form of operators $\Phi(x)$ and $\Pi(x)$ and can therefore be diagonalized and reduced to an infinite set of oscillators. The only problem is to take into consideration the constraints that are imposed on $\Phi(x)$ and $\Pi(x)$. From now on we can work in the center-of-mass system by setting $v = 0$, or $I = 0$. This simplifies the expression for H_2 , indeed:

$$H_2 = \int_0^L dx \left\{ \frac{1}{2}\Pi^2(x) + \frac{1}{2}\left(\frac{\partial\Phi(x)}{\partial x}\right)^2 + \mu^2(3\phi_0^2(x) - 1)\Phi^2(x) \right\}, \quad (36)$$

Let the functions $V_n(x)$ be an orthonormal set of solutions of the equation

$$\left\{ -\frac{d^2}{dx^2} + \mu^2\left(\frac{6k^2}{1+k^2}\text{sn}^2\left(\sqrt{\frac{2}{1+k^2}}, k\mu x\right) - 1\right) \right\} V_n(x) = \mathcal{E}_n^2 V_n(x). \quad (37)$$

We now consider the expansions of operators $\Phi(x)$ and $\Pi(x)$ in terms of $V_n(x)$:

$$\Phi(x) = \sum_n' \sqrt{\frac{1}{2\mathcal{E}_n}} [a_n V_n(x) + a_n^+ V_n^*(x)], \quad (38)$$

$$\Pi(x) = i \sum_n' \sqrt{\frac{\mathcal{E}_n}{2}} [a_n^+ V_n^*(x) + a_n V_n(x)], \quad (39)$$

in which the prime denotes that the sum does not contain the modes with zero energy (translational mode). This system of functions fulfills the constraint

$$\int_0^L dx N(x) V_n(x) = 0. \quad (40)$$

We may choose without loss of generality $N(x) = m \frac{\partial\phi_0(x)}{\partial x}$. Furthermore, the condition of completeness of the functions $V_n(x)$ is:

$$\sum_n [V_n(x) V_n^*(y) + V_n(y) V_n^*(x)] = A(x, y) \quad (41)$$

which indicates, that this system of functions does not contain the function corresponding to the zero mode.

The creation and annihilation operators a_n^+ and a_n obey the commutation relation

$$[a_n, a_n^+] = 1. \quad (42)$$

After some notations, equation (37) can be reduced to the following form.

$$\frac{d^2 V_n(z)}{dz^2} + [\lambda + N(N+1)k^2 \mathbf{sn}^2(z, k)] V_n(z) = 0, \quad (43)$$

which is a Lamé equation. The notations we have introduced are:

$$z = \sqrt{\frac{2}{1+k^2}} \mu x, \quad \lambda = \frac{(\mathcal{E}_n^2 + 2\mu^2)\sqrt{1+k^2}}{\sqrt{2}} \quad (44)$$

In our case of double-well potential $N=2$. Periodic solutions of the Lamé equation are well studied[11]. All the eigenvalues of the Lamé equation are discrete and corresponding solutions are called Lamé polynomials. There are in general $2N+1$ discrete eigenvalues for given N and 6 in our case, one of them is zero, which is excluded from the set of functions $V_n(x)$. This is also confirmed by the completeness condition of the functions $V_n(x)$. It is important, that among these eigenvalues only one is negative (let it denote by \mathcal{E}_{-1}), namely

$$\mathcal{E}_{-1}^2 = 2\mu^2 \left(1 - 2 \frac{\sqrt{1-k^2(1-k^2)}}{1+k^2}\right) < 0. \quad (45)$$

All other eigenvalues are positive. On substituting the expansions (38) and (39) into (36) we obtain the diagonal form of H_2 :

$$H_2 = \frac{1}{2} \sum_n' \mathcal{E}_n [a_n^+ a_n + a_n a^+]. \quad (46)$$

The energy of the lowest state (we name it quantum periodic instanton) $E_2 = \frac{1}{2} \sum_n' \mathcal{E}_n$ diverges, but this is beyond the scope of our interest, since we want to estimate the lifetime of our physical system (one can find the details of regularization in[8]). Obviously, the energy of the system is a complex quantity. The imaginary part of the energy $\Im E = \frac{\Gamma}{2}$ (with Γ being a decay width) is a measure for the lifetime t_l of the system:

$$t_l = \frac{2}{\Gamma} = \frac{\sqrt{2}}{\mu \sqrt{\left(2 \frac{\sqrt{1-k^2(1-k^2)}}{1+k^2} - 1\right)}} \quad (47)$$

In the limit $k \rightarrow 1$ ($L \rightarrow \infty$) the lifetime t_l approaches the ∞ , as it should be for a stable field configuration. In the opposite limit of $k \rightarrow 0$ the lifetime $t_l = \frac{\sqrt{2}}{\mu}$ is a finite quantity. This is the lifetime of a sphaleron, a configuration at the top of the potential barrier.

Acknowledgments

This article is carried out under the project supported by the Grant of Georgian National Science Foundation GNSF/ST-08/4-405.

References

- [1] S. Coleman, in: The ways of subnuclear Physics, ed. A. Zichichi (Plenum, New York, 1979) p.805.
- [2] N. S. Manton, *Phys.Rev.* **D28**, 2019, (1983).
- [3] F. Klinkhamer and N. S. Manton, *Phys.Rev.* **D30**, 2212, (1984).
- [4] N. S. Manton and T. M. Samols, *Phys. Lett* **B207**, 179, (1988).
- [5] Jiu-Qing Liang, H. J. W. Müller-Kirsten and D. H. Tchrakian, *Phys. Lett.* **B282**, 105, (1992).
- [6] M. Sakamoto, M. Tachibana and K. Takenaga, *Phys.Lett.* **B457**,33,(1999)
- [7] G. Mussardo, V. Riva, G. Sotkov and G. Delfino, *Nucl. Phys.* **B736**, 259, (2006).
- [8] M. Pawellek *J. Phys. A: Math. Theor.* **42**, 045404, (2009).
- [9] M. Pawellek *Nucl.Phys.***B810**, 527,(2009).
- [10] A. Shurgaia and H. J. W. Müller-Kirsten, *Int. J. Mod. Phys.* **A22**, 3655, (2007).
- [11] F. M. Arscott, *Periodic differential equations*, (Pergamon, Oxford, 19640).

Quantum Properties of Periodic Instantons on a Circle

A. V. SHURGAIA¹

Department of Theoretical Physics
A. Razmadze Mathematical Institute
1 M. Alexidze str. 0193 Tbilisi
Georgia

Abstract

Quantum properties of a (1+1)-dimensional scalar theory on a cylinder with a compact spatial part, namely, $0 \leq x \leq L$, are considered. In particular, quantum theory around the classical periodic field configurations is studied and the lifetime of a quantum periodic instanton is estimated.

1. Periodic field configurations known as the periodic instantons have been for many years in the field of view of researchers in view of their important role in the process of quantum tunneling and the associated phase transitions. The attractiveness of such field objects is that, depending on the total energy they interpolate between the vacuum and saddle point states (sphalerons) and thus have the ability to be responsible for the phase transition from the classical processes at high energies to quantum tunneling ones at low energies. The complexity of actual physical theories does not allow to obtain exact results, hence lower dimensional models which make it possible to perform some analytical calculations are of interest. From this point of view (1+1)-dimensional scalar theory is subjected to intensive study. Along with stable field configurations of finite energy more recently unstable configurations like bounces[1], sphalerons[2, 3], periodic instantons[4, 5] have been investigated. Recently, interest has grown in configurations on compact space and configurations like (anti)periodic (called also twisted fields) are in addition investigated[6]-[9]. In the cited papers the exact solutions of classical equations of motion on a circle, as well as the equation of fluctuations are investigated and the regularization of the energy in one-loop approximation is carried out. In the present paper we are interested in periodic classical field configurations and their quantum properties in two-dimensional scalar theory, the spatial part of which is a circle with circumference L . There is an interesting interpretation of nontrivial classical field configurations for finite L , provided, that the potential of the model we explore has two distinct vacua separated by a finite potential barrier. This allows to consider the process of creating of kink-antikink pair at some energy. They

¹e-mail: avsh@rmi.acnet.ge

propagate along the circle in opposite directions and annihilate at the meeting points leaving the field in the other vacuum. The largest distance between kink and antikink corresponds to the state at the top of a potential barrier - that is the saddle point of the energy functional. With increasing L unstable configurations of many kink-antikink states can be appeared[4]. In the present paper we start from the quantum theory in Schrödinger representation. We use the symmetry properties of the theory and construct a perturbation theory in the inverse powers of a coupling constant using the collective coordinates method (see for general theory and some application[10] and references therein). We calculate the quantum energy levels and estimate the lifetime of the system.

2. Thus, we consider the system with the lagrangian density

$$\mathcal{L} = \frac{1}{2} \left(\frac{\partial \phi(t, x)}{\partial x_\mu} \right)^2 - U(\phi(t, x), g) \quad (1)$$

with the potential satisfying

$$U(\phi(t, x), g) = \frac{1}{g^2} U(g\phi(t, x), 1)$$

in which $0 \leq x \leq L$. We work in the Schrödinger representation and look for solutions of the equation:

$$(H - E)\Psi(\phi(x), \pi(x)) = 0. \quad (2)$$

The corresponding quantum Hamiltonian is

$$H = \int_0^L dx \left\{ \frac{1}{2} \pi(x)^2 + \frac{1}{2} \left(\frac{\partial \phi(x)}{\partial x} \right)^2 + U(\phi(x), g) \right\} \quad (3)$$

subject to the following commutation relation:

$$[\phi(x), \pi(y)] = i\delta(x - y). \quad (4)$$

We now develop a perturbation theory for the double-well potential $U(\phi(x), g)$:

$$U(\phi(x), g) = \frac{\mu^2}{2g^2} (\phi(x)^2 - g^2)^2. \quad (5)$$

The system possesses a translational invariance and corresponding conserved quantity is the momentum of the system:

$$P = - \int_0^L dx \frac{\partial \phi(x)}{\partial x} \pi(x) \quad (6)$$

Let $\phi_0(x)$ be a c-number and consider the following transformation of the field $\phi(x)$:

$$\phi(x) = g\phi_0(x - q) + \Phi(x - q). \quad (7)$$

The parameter q , being called a collective coordinate, together with the new field $\Phi(x)$ form now a new set of variables (operators) of the system: $\{\phi(x), \pi(x)\} \rightarrow \{q, p_q, \Phi(x), \tilde{\Pi}(x)\}$. Thereby the phase space has been expanded and in order to keep the number of independent variables unchanged the additional condition must be imposed (actually this is a gauge condition):

$$\int_0^L dx N(x) \Phi(x) = 0, \quad (8)$$

in which $N(x)$ is an arbitrary function. Really we need one more additional condition, but as we will see below this condition will appear as a constraint in the extended space. One can always choose $N(x)$ such that the relation

$$\int_0^L dx N(x) \frac{\partial \phi_0(x)}{\partial x} = 1. \quad (9)$$

holds. The operator $\pi(x)$ is considered as a functional derivative $-i\delta/\delta\phi(x)$ in a Hilbert space spanned by the vectors $|\Psi\rangle$ and hence should be expressed through the set of new variables. For this we need to define the projection operator $A(x, y) = \delta(x - y) - \frac{\partial \phi_0(x)}{\partial x} N(y)$, with the properties

$$\int_0^L dy A(x, y) \frac{\partial \phi_0(y)}{\partial x} = \int_0^L dx N(x) A(x, y) = 0.$$

Now it is easy to show that the operator $\pi(x)$ can be rewritten as follows:

$$\pi(x) = \Pi(x - q) - \frac{N(x - q)}{g(1 + \frac{1}{g}F)} \{p_q + M(\Phi, \Pi)\}, \quad (10)$$

in which the quantities F and $M(\Phi, \Pi)$ are defined by the following equalities:

$$F = \int_0^L dx N(x) \frac{\partial \Phi(x)}{\partial x}, \quad M(\Phi, \Pi) = \int_0^L dx \frac{\partial \Phi(x)}{\partial x} \Pi(x). \quad (11)$$

Besides the operator $\Pi(x)$, determined as

$$\Pi(x) = \int_0^L dy A(y, x) \frac{\delta}{i\delta\Phi(y)}, \quad (12)$$

satisfies the condition

$$\int_0^L dx \frac{\partial \phi_0(x)}{\partial x} \Pi(x) = 0, \quad (13)$$

that is the constraint of theory (that we have mentioned above) being considered as a system with constraints. The operators q and $p_q = \frac{\partial}{i\partial q}$ are conjugate to each other such that $[q, p_q] = i$. Thus all extra degrees of freedom are now fixed. The operator $\Pi(x)$ obeys the following commutation relation:

$$[\Phi(x), \Pi(y)] = iA(x, y). \quad (14)$$

One can now check by substituting (10) into (6) that the momentum operator of the system coincides with p_q and consequently $P = p_q$. On substituting (7) and (10) into (3) one obtains the Hamiltonian in the new phase space. We see, that $\pi(x)$ depends on q . In order to obtain the Hamiltonian we have to integrate over x in the range from 0 to L . Shifting the integration variable x changes the limits of integration from $-q$ to $L - q$. Nevertheless there is no explicit dependence of the Hamiltonian from q . Taking into consideration the periodicity of the phase space variables and the Leibnitz integral rule, it is easy to verify that $\frac{dH}{dq} = 0$. This allows us to extract the q -dependence of the wave function of the system $\Psi(\Phi(x), q)$. Before doing so we need to transform the wave function, indeed

$$\Psi(\Phi, q) = e^{ig \int_0^L dx s(x) \Phi(x)} \Psi'(\Phi(x), q). \quad (15)$$

which means that the momentum operator $\Pi(x)$ has to be replaced by $gs(x) + \Phi(x)$. The c-number $s(x)$ is subject to the same constraint as the operator $\Pi(x)$, namely $\int_0^L dx \frac{\partial \phi_0(x)}{\partial x} s(x) = 0$. If this condition is not being fulfilled then one defines with the help of the projection operator a new quantity $s'(x)$ that obeys it. Thus the equality (10) may be rewritten as follows:

$$\pi(x) = \Pi(x - q) - \frac{N(x - q)}{g(1 + \frac{1}{g}F)} \{p_q + gM(\Phi, s) + M(\Phi, \Pi)\}. \quad (16)$$

3. After all these one can start to solve the Schrödinger equation $(H - E)\Psi'(\Phi(x), q) = 0$. As we have seen above the Hamiltonian H does not explicitly depend on q and therefore one may factorize the wave function $\Psi'(\Phi(x), q)$ as follows:

$$\Psi'(\Phi(x), q) = e^{ig^2 I q} \Psi''(\Phi(x)). \quad (17)$$

Thus the operator p_q should be replaced by the c-number $g^2 I$. We will solve the Schrödinger equation by using a perturbation theory and consequently we need now to expand the Hamiltonian in series in inverse powers of g . Energy and the wave function are also to expand in appropriate series:

$$H = g^2 H_0 + g H_1 + H_2 + g^{-1} H_3 + \dots, \quad (18)$$

$$E = g^2 E_0 + g E_1 + E_2 + g^{-1} E_3 + \dots, \quad (19)$$

$$\Psi''(\Phi(x)) = \Psi_0 + g^{-1} \Psi_1 + \dots \quad (20)$$

So we should solve the system of equations:

$$(H_0 - E_0)\Psi_0 = 0, \quad (21)$$

$$(H_0 - E_0)\Psi_1 + (H_1 - E_1)\Psi_0 = 0, \quad (22)$$

$$((H_0 - E_0)\Psi_2 + (H_1 - E_1)\Psi_1 + (H_2 - E_2)\Psi_0 = 0. \quad (23)$$

.....

Leading term of the Hamiltonian does not contain field operators and therefore the equation in corresponding approach $(H_0 - E_0)\Psi_0 = 0$ is obeyed identically

if

$$E_0 = H_0 = \int_0^L dx \left\{ \frac{1}{2} \left(\frac{\partial \phi_0(x)}{\partial x} \right)^2 + U(\phi(x), 1) + \frac{1}{2} (s(x) - IN(x))^2 \right\}. \quad (24)$$

The next approximation of the system of equations we consider is:

$$(\Psi_0, (H_1 - E_1)\Psi_0) = 0, \quad (25)$$

which is linear in field operators. The regularity of the function Ψ_0 requires H_1 and E_1 to be identically zero. This is accomplished by using the additional conditions imposed on the field operators, and if required

$$s(x) - IN(x) = -v \frac{\partial \phi_0(x)}{\partial x}, \quad (26)$$

and

$$-(1 - v^2) \frac{\partial^2 \phi_0(x)}{\partial x^2} + U'(\phi_0(x), 1) = 0 \quad (27)$$

This is a pure classical equation of motion, the solution of which under periodic boundary condition for the potential U of interest is:

$$\phi_0(x) = \sqrt{\frac{2k^2}{1+k^2}} \text{sn}\left(\sqrt{\frac{2}{1+k^2}} \frac{\mu x}{\sqrt{1-v^2}}, k\right), \quad (28)$$

$$(29)$$

in which $0 \leq k \leq 1$ is the modulus of elliptic integrals. The periodicity condition $\phi_0(x + L) = \phi_0(x)$ implies that

$$L = 4n\mathbf{K}(k) \sqrt{\frac{1+k^2}{2}} \frac{\sqrt{1-v^2}}{\mu}, \quad (30)$$

$$(31)$$

with n integer. So, there are critical values of the quantity L , at which there are bifurcations of the stationary points of the energy E_0 . These solutions interpolate between the stable field configurations as k^2 tends to 1 (with L approaching the ∞) and the unstable ones (at the top of the potential barrier) for $k^2 \rightarrow 0$, latter being called sphalerons[4, 5]).

Let us now turn to the zero energy E_0 , that takes the following form:

$$E_0 = \int_0^L dx \left\{ \frac{1}{2} (1 + v^2) \left(\frac{\partial \phi_0(x)}{\partial x} \right)^2 + U(\phi(x), 1) \right\} \quad (32)$$

and is for (5):

$$E_0 = \frac{n\sqrt{2}\mu\{8(1+k^2)\mathbf{E}(k) - [(1-k^2)(5+3k^2) + 3v^2(1-k^2)]\mathbf{K}(k)\}}{3(1+k)^{3/2}\sqrt{1-v^2}}, \quad (33)$$

which for $v = 0$ is consistent with the expression obtained in [4]. Using the equality (26) and the condition that is imposed on $s(x)$ one can obtain for the momentum I :

$$I = \frac{8n\sqrt{2}\mu[(1+k^2)\mathbf{E}(k) - (1-k^2)\mathbf{K}(k)]}{3(1+k^2)^{3/2}} \frac{v}{\sqrt{1-v^2}}. \quad (34)$$

It is easy to verify that v is a velocity of the center-of-mass of the system.

4. Let us now proceed with the study of quantum correction to the ground state energy, for which the equation

$$(H_2 - E_2)\Psi_0 = 0 \quad (35)$$

has to be solved. The operator H_2 is quadratic form of operators $\Phi(x)$ and $\Pi(x)$ and can therefore be diagonalized and reduced to an infinite set of oscillators. The only problem is to take into consideration the constraints that are imposed on $\Phi(x)$ and $\Pi(x)$. From now on we can work in the center-of-mass system by setting $v = 0$, or $I = 0$. This simplifies the expression for H_2 , indeed:

$$H_2 = \int_0^L dx \left\{ \frac{1}{2} \Pi^2(x) + \frac{1}{2} \left(\frac{\partial \Phi(x)}{\partial x} \right)^2 + \mu^2 (3\phi_0^2(x) - 1) \Phi^2(x) \right\}, \quad (36)$$

Let the functions $V_n(x)$ be an orthonormal set of solutions of the equation

$$\left\{ -\frac{d^2}{dx^2} + \mu^2 \left(\frac{6k^2}{1+k^2} \text{sn}^2 \left(\sqrt{\frac{2}{1+k^2}}, k\mu x \right) - 1 \right) \right\} V_n(x) = \mathcal{E}_n^2 V_n(x). \quad (37)$$

We now consider the expansions of operators $\Phi(x)$ and $\Pi(x)$ in terms of $V_n(x)$:

$$\Phi(x) = \sum_n' \sqrt{\frac{1}{2\mathcal{E}_n}} [a_n V_n(x) + a_n^+ V_n^*(x)], \quad (38)$$

$$\Pi(x) = i \sum_n' \sqrt{\frac{\mathcal{E}_n}{2}} [a_n^+ V_n^*(x) + a_n V_n(x)], \quad (39)$$

in which the prime denotes that the sum does not contain the modes with zero energy (translational mode). This system of functions fulfills the constraint

$$\int_0^L dx N(x) V_n(x) = 0. \quad (40)$$

We may choose without loss of generality $N(x) = m \frac{\partial \phi_0(x)}{\partial x}$. Furthermore, the condition of completeness of the functions $V_n(x)$ is:

$$\sum_n [V_n(x) V_n^*(y) + V_n(y) V_n^*(x)] = A(x, y) \quad (41)$$

which indicates, that this system of functions does not contain the function corresponding to the zero mode.

The creation and annihilation operators a_n^+ and a_n obey the commutation relation

$$[a_n, a_n^+] = 1. \quad (42)$$

After some notations, equation (37) can be reduced to the following form.

$$\frac{d^2 V_n(z)}{dz^2} + [\lambda + N(N+1)k^2 \mathbf{sn}^2(z, k)] V_n(z) = 0, \quad (43)$$

which is a Lamé equation. The notations we have introduced are:

$$z = \sqrt{\frac{2}{1+k^2}} \mu x, \quad \lambda = \frac{(\mathcal{E}_n^2 + 2\mu^2)\sqrt{1+k^2}}{\sqrt{2}} \quad (44)$$

In our case of double-well potential $N=2$. Periodic solutions of the Lamé equation are well studied[11]. All the eigenvalues of the Lamé equation are discrete and corresponding solutions are called Lamé polynomials. There are in general $2N+1$ discrete eigenvalues for given N and 6 in our case, one of them is zero, which is excluded from the set of functions $V_n(x)$. This is also confirmed by the completeness condition of the functions $V_n(x)$. It is important, that among these eigenvalues only one is negative (let it denote by \mathcal{E}_{-1}), namely

$$\mathcal{E}_{-1}^2 = 2\mu^2 \left(1 - 2 \frac{\sqrt{1-k^2(1-k^2)}}{1+k^2}\right) < 0. \quad (45)$$

All other eigenvalues are positive. On substituting the expansions (38) and (39) into (36) we obtain the diagonal form of H_2 :

$$H_2 = \frac{1}{2} \sum_n' \mathcal{E}_n [a_n^+ a_n + a_n a_n^+]. \quad (46)$$

The energy of the lowest state (we name it quantum periodic instanton) $E_2 = \frac{1}{2} \sum_n' \mathcal{E}_n$ diverges, but this is beyond the scope of our interest, since we want to estimate the lifetime of our physical system (one can find the details of regularization in[8]). Obviously, the energy of the system is a complex quantity. The imaginary part of the energy $\Im E = \frac{\Gamma}{2}$ (with Γ being a decay width) is a measure for the lifetime t_l of the system:

$$t_l = \frac{1}{2\Im E} = \frac{1}{2\mu \sqrt{2 \left(2 \frac{\sqrt{1-k^2(1-k^2)}}{1+k^2} - 1\right)}} \quad (47)$$

In the limit $k \rightarrow 1$ ($L \rightarrow \infty$) the lifetime t_l approaches the ∞ , as it should be for a stable field configuration. In the opposite limit of $k \rightarrow 0$ the lifetime $t_l = \frac{\sqrt{2}}{\mu}$ is a finite quantity. This is the lifetime of a sphaleron, a configuration at the top of the potential barrier.

5. Let us discuss obtained results. First of all note, that the classical equation for the field $\phi(t, x)$ is invariant under Lorentz transformation. Using translational invariance in time and space makes it possible to consider the field as $\phi(x - vt)$ such that the equation reduces to

$$-(1 - v^2) \frac{d^2}{d\phi(z)^2} + U'(\phi(z)) = 0 \quad (48)$$

with $z = x - vt$, which is exactly the equation (27). Nevertheless we have seen that the Lorentz invariance has been violated. We are convinced that this invariance is restored when $k = 1$. Indeed the solution (28) becomes:

$$\phi(x) = \tanh\left(\frac{\mu x}{\sqrt{1 - v^2}}\right), \quad (49)$$

which describes a moving kink. The period L tends to the infinity (a line). The energy and the momentum take the the form:

$$E_0 = \frac{8\mu n}{3\sqrt{1 - v^2}}, \quad I = \frac{8\mu n v}{3\sqrt{1 - v^2}}. \quad (50)$$

This quantities can be rewritten as

$$E_0 = 2nE_k, \quad I = 2nI_k \quad (51)$$

with $E_k = 4\mu n/3\sqrt{1 - v^2}$ and $I_k = 4\mu n v/3\sqrt{1 - v^2}$ being the energy and the momentum of a kink. Thus in the limit of an infinite line the solution (28) describes a kink, while the energy and the momentum correspond to the system of $2n$ kink-antikink. The Lorentz invariance is evident.

When $k = 0$ the field $\phi(x) = 0$ and matches the top of the potential barrier - this is a sphaleron². The period $L = L_n = \sqrt{2\pi n}\sqrt{1 - v^2}/\mu$. One need to emphasize that the circumference L is subjected to the contraction: the larger the velocity v the smaller L . The momentum I for $k = 0$ is zero. Consistency of the equation of motion with periodic boundary conditions requires velocity to be zero - $v = 0$. This means that the energy in (33) becomes $E_0 = \frac{\sqrt{2}\mu\pi n}{2}$.

Let us now approximate the energy and the momentum for two limiting cases of k , namely for a) $k \ll 1$ and b) $k \lesssim 1$ or equivalently $k' = \sqrt{1 - k^2} \ll 1$. The standard expansions of complete elliptic functions in k and k' allows us to approximate the energy and the momentum for any fixed v as follows:

$$\begin{aligned} E_0 &= \frac{\sqrt{2}\mu\pi n}{2} \sqrt{1 - v^2} + \frac{\sqrt{2}\mu\pi n}{8\sqrt{1 - v^2}} \left[3k^2 - \frac{87}{16}k^4 + v^2(13k^2 - \frac{329}{16}k^4) \right] \\ I &= \frac{2\sqrt{2}\mu\pi n v}{\sqrt{1 - v^2}} \left(k^2 - \frac{13}{8}k^4 \right), \\ k &\ll 1. \end{aligned} \quad (53)$$

²We prefer to retain the term "periodic instanton" for periodic field configurations, that is often used in the literature[12] and references there)

For completeness we give for $k \ll 1$ the expressions of the solution (28) and the lifetime:

$$\phi_0(x) = k\sqrt{2}\sin\frac{\sqrt{2}\mu x}{\sqrt{1-v^2}}, \quad (54)$$

$$t_l = \frac{\sqrt{2}}{4\mu} + \frac{3\sqrt{2}}{8\mu}\sqrt{1-v^2}(k^2 + k^4). \quad (55)$$

It is also interesting to rewrite these quantities in terms of circumference L . These are:

$$E_0 = \frac{\sqrt{2}\mu n\pi}{2}\sqrt{1-v^2} + \frac{\sqrt{2}\mu n\pi}{2\sqrt{1-v^2}} \left\{ \frac{L-L_n}{L_n} - \frac{87}{36} \frac{(L-L_n)^2}{L_n^2} + v^2 \left[\frac{13}{3} \frac{L-L_n}{L_n} - \frac{329}{36} \frac{(L-L_n)^2}{L_n^2} \right] \right\}, \quad (56)$$

$$I = \frac{8\sqrt{2}\mu\pi n v}{3\sqrt{1-v^2}} \left\{ \frac{L-L_n}{L_n} - \frac{13}{6} \frac{(L-L_n)^2}{L_n^2} \right\}, \quad (57)$$

$$t_l = \frac{\sqrt{2}}{4\mu} + \frac{\sqrt{2}}{2\mu} \left\{ \frac{L-L_n}{L_n} + \frac{4}{3} \frac{(L-L_n)^2}{L_n^2} \right\}. \quad (58)$$

The expression for the energy at $v = 0$ differs from that one obtained in[4], namely the coefficient of the second term in (56) is $87/36$ whereas in the cited paper is $8/3$. This difference can be explained by assumption made in[4], namely because of periodicity of $\phi_0(x)$ the derivative $\partial\phi_0(x)/\partial x$ has been chosen to be zero at $x = 0$ and L . We did not make such assumption. Correspondingly, they contribute to the expression for energy. It should be noted that the coincidence of the coefficients of the first term takes place due to the fact that when $L = L_n$ the energy (56) must correspond to the sphaleron. This happens at $L = L_n$ (and $v = 0$). One sees that the formulae (56)-(58) correspond to those ones at $L = L_n$ (and $v = 0$).

We now turn to the opposite limit $k \rightarrow 1$ or $k' = \sqrt{1-k^2} \rightarrow 0$. Using the expansion of $\mathbf{K}(k)$ one gets for k' :

$$k'^2 = 16 \exp\left(-\frac{L\mu}{2n\sqrt{1-v^2}}\right)$$

expanding the energy, the momentum and the lifetime in k' up to k'^4 gives the following results:

$$E_0 = \frac{8\mu n}{3\sqrt{1-v^2}} - \frac{32\mu n}{\sqrt{1-v^2}} \left(1 + \frac{v^2 L\mu}{n\sqrt{1-v^2}}\right) \exp\left(-\frac{L\mu}{n\sqrt{1-v^2}}\right), \quad (59)$$

$$I = \frac{8\mu n v}{3\sqrt{1-v^2}} - \frac{32\mu n v}{\sqrt{1-v^2}} \exp\left(-\frac{L\mu}{n\sqrt{1-v^2}}\right), \quad (60)$$

$$t_l = \frac{\sqrt{3}}{48\mu} \left(1 - 8 \exp\left(-\frac{L\mu}{2n\sqrt{1-v^2}}\right)\right) \exp\left(\frac{L\mu}{2n\sqrt{1-v^2}}\right). \quad (61)$$

These formulae describe the system near the vacuum. One sees again that the Lorentz relation between the energy and the momentum holds for $L \rightarrow \infty$. Furthermore the energy (59) receives the additional term as compared to the formula (15b) in [4], namely second term, that describes the interaction energy between kink and antikink in addition to the factor $\sqrt{1-v^2}$ now depends on velocity v (in parentheses in front of the exponential). In a similar manner in formula (56) additional term appears that depends on v . One could rewrite these expansions for $L \gtrsim L_n$ and $L \gg L_n$ in the nonrelativistic limit, but we do not give the corresponding expressions since they are too long. We only mention that there is no known relations between the energy and the momentum of the system. They will be restored in the infinite line limit. Namely:

$$E_0 = \frac{8n\mu}{3} + \frac{8n\mu}{3} \frac{v^2}{2}, \quad (62)$$

$$I = \frac{8n\mu}{3} v, \quad (63)$$

for $L \rightarrow \infty$ and $v \ll 1$.

The lifetime is evaluated for the center-of-mass of the system. The formula (58) gives the lifetime near the top of the potential barrier and in case $L = L_n$ one obtains a finite value. In the opposite limit $L \rightarrow \infty$ the dominant term in (61) is the exponential $\exp(\frac{L\mu}{2n\sqrt{1-v^2}})$, which is growing up infinitely as $L \rightarrow \infty$.

Acknowledgments

This article is carried out under the project supported by the Grant of the Georgian National Science Foundation GNSF/ST-08/4-405.

References

- [1] S. Coleman, in: The ways of subnuclear Physics, ed. A. Zichichi (Plenum, New York, 1979) p.805.
- [2] N. S. Manton, *Phe.Rev.* **D28**, 2019, (1983).
- [3] F. Klinkhamer and N. S. Manton, *Phys.Rev.* **D30**, 2212, (1984).
- [4] N. S. Manton and T. M. Samols, *Phys. Lett* **B207**, 179, (1988).
- [5] Jiu-Qing Liang, H. J. W. Müller-Kirsten and D. H. Tchrakian, *Phys. Lett.* **B282**, 105, (1992).
- [6] M. Sakamoto, M. Tachibana and K. Takenaga, *Phys.Lett.* **B457**,33,(1999)
- [7] G. Mussardo, V. Riva, G. Sotkov and G. Delfino, *Nucl. Phys.* **B736**, 259, (2006).

- [8] M. Pawellek *J. Phys. A: Math. Theor.* **42**, 045404, (2009).
- [9] M. Pawellek *Nucl.Phys.***B810**, 527,(2009).
- [10] A. Shurgaia *Theor.Math.Phys.*, 45, 873, (1980), *Teor.Mat.Fiz.* 45, 46, (1980).
A. Shurgaia *Theor.Math.Phys.*, 57, 1216, (1984), *Teor.Mat.Fiz.* 57, 392, (1983).
A. Shurgaia and H. J. W. Müller-Kirsten, *Int. J. Mod. Phys. A***22**, 3655, (2007).
- [11] F. M. Arscott, *Periodic differential equations*, (Pergamon, Oxford, 1964).
- [12] H. J. W. Müller-Kirsten, *Introduction to quantum mechanics*, (World Scientific, Singapore, 2006).